$\boxed{4.1}$ Entropy function is given by:

$$H(p) = -\sum_{i=1}^{x} p_i \log p_i$$

1- Continuity: Entropy function is continous because logarithm function $\log p_i$ is continous for $p_i > 0$.

2. Non-negativity: Since p_i is a probability, it satisfies $0 \leq p_i \leq 1$. Moreover, function $-p_i \log p_i$ is always non-negative because $\log p_i \leq 0$ for $0 < p_i \leq 1$.

3- Boundaries: The entropy reaches a maximum when all outcomes are equally probable.

$$H_{max} = \log x$$

Since $H(p)$ is always less than or equal to $\log x$, it is bounded.

4- ~~Independance (Additivity)~~

$\boxed{4.2}$ Mutual Information:

$$I(x,y) = H(x) + H(y) - H(x,y)$$

using the definition of entropy

$$H(x,y) = -\sum_x \sum_y p(x,y) \log p(x,y)$$

Re-writing it using conditional probability

$$H(x,y) = -\sum_x \sum_y p(x,y) \log [p(x|y)\, p(y)]$$

Expanding:

$$H(x,y) = -\sum_x \sum_y p(x,y) \log p(x|y) - \sum_x \sum_y p(x,y) \log p(y)$$

Since $\sum_x p(x,y) = p(y)$

$$H(x,y) = H(x|y) + H(y)$$

Rearranging:

$$H(y) - H(y|x) = H(x) + H(y) - H(x,y)$$

which is the definition of mutual information

$$I(x,y) = H(x) - H(x|y) = H(y) - H(y|x)$$

4.3 using binomial Theorem

a)

$$\varepsilon_1 = B(2, \varepsilon_0, 3) + B(3, \varepsilon_0, 3)$$
$$= \binom{3}{2} \varepsilon_0^2 (1-\varepsilon_0) + \binom{3}{3} \varepsilon_0^3$$
$$= 3 \varepsilon_0^2 (1-\varepsilon_0) + \varepsilon_0^3$$
$$= 3 \varepsilon_0^2 - 2 \varepsilon_0^3$$

b) Each group of 3 bits producers output with error probability $3\varepsilon_0^2 - 2\varepsilon_0^3$. This becomes new ε_0' for next level.

Final probability = $3(3\varepsilon_0^2 - 2\varepsilon_0^3)^2 - 2(3\varepsilon_0^2 - 2\varepsilon_0^3)^3$

c) For N iterations, number of bits needed = 3^N

Error probability follows recursine formula:

$$\varepsilon_n = 3(\varepsilon_n - 1)^2 - 2(\varepsilon_n - 1)^3$$

For small ε, higher powers become negligible so error scales approximatelly as:

$$\varepsilon^{(2N)}$$

$\boxed{4.4}$ $$H(N(\mu,\sigma^2)) = -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}} \log\left(\frac{1}{2\pi\sigma^2} e^{\frac{-x^2}{2\sigma^2}}\right) dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-x^2}{2\sigma^2}} \left(-\frac{1}{2}\log(2\pi\sigma^2) \frac{-x^2}{2\sigma^2}\right) dx$$

$$* \quad \because \frac{d}{dx} e^{\frac{-x^2}{2\sigma^2}} = -\frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}$$

$$= \frac{1}{2}\log(2\pi\sigma^2) \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} \frac{x^2}{2\sigma^2} e^{\frac{x^2}{2\sigma^2}} dx$$

$$= \frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{x}{\sqrt{2\pi\sigma^2}}\right)\left(\frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}\right) dx$$

$$= \frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dx$$

$$= \frac{1}{2}\log(2\pi\sigma^2) + \frac{1}{2}$$

$$= \frac{1}{2}\log(2\pi e\sigma^2)$$

4.5

A signal to noise ratio of 20dB means,

$$20 = 10 \log_{10} (S/N)$$

So, $\frac{S}{N} = 10^2 = 100$

Band width : $\Delta f = 3300$ Hz
(Telephone Line)

Capacity Calculation:

Shannon's capacity formula:

$$C = \Delta f \log_2 (1 + S/N)$$

$$C = 3300 \log_2 (1 + 100)$$

$$C = 3300 \times \log_2 (101)$$

$$C = 22 \text{ kbit/s}$$

For 1 Gbit/s

We solve for S/N in:

$$1 \times 10^9 = 3300 \log_2 (1 + S/N)$$

$$\frac{10^9}{3300} = \log_2 (1 + S/N)$$

$$2^{10^9/3300} - 1 = S/N$$

Since this is a large number, we use logarithms:

$$S/N_{dB} = \frac{10^9}{3300} \times 10 \log_{10} (2)$$

$$= 9 \times 10^5 \text{ dB}$$

This is unrealistically high, making 1 Gbit/s impractical.

4.6 To estimate the mean of Gaussian distribution:

$$\hat{x}_0 = \frac{1}{n} \sum_{i=1}^{n} x_i$$

To check if estimator in unbiased, lets calculate the expected value of $\hat{x}_0$:

$$E[\hat{x}_0] = E\left[\frac{1}{n} \sum_{i=1}^{n} x_i\right]$$

Since expectation distribute over addition:

$$[\hat{x}_0] = \frac{1}{n} \sum_{i=1}^{n} E[x_i]$$

Since each x_i comes from Gaussian distribution with mean x_0:

$$E[x_i] = x_0$$

So, $$E[\hat{x}_0] = \frac{1}{n} \sum_{i=1}^{n} x_0 = \frac{n x_0}{n} = x_0$$

This shows sample mean is an unbiased estimator of x_0.

Now, from Cramér-Rao Bound:

variance of any unbiased estimator $\geq$

$$\frac{1}{\text{Fisher information}}$$

Estimator can never have an error smaller than this bound.

For a Gaussian distribution with variance σ^2, Fisher Information is:

$$J(x_0) = \frac{n}{\sigma^2}$$

Applying the Cramér_Rao Bound

$$var(\hat{x}_0) \geq 1/J(x_0)$$

Substituting $J(x_0) = n/\sigma^2$

$$var(\hat{x}_0) \geq \frac{1}{n/\sigma^2} = \frac{\sigma^2}{n}$$

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